CS-151 Quantum Computer Science: Problem Set 1

Professor: Saeed Mehraban
TA: Dale Jacobs

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Guidelines: ***The deadline to return this problem set is 11.59pm on Monday, January 29***. *Please show all work for full credit. Remember that you can collaborate with each other in the preliminary stages of your progress, but each of you must write your solutions independently. Submission of the problem set should be via Gradescope only. Best wishes!*

Problem 1 (15 points) *Let* $z = \frac{\sqrt{3}}{2} + \frac{1}{2}i$.

a) *Find* z^2. *What are* $\Re(z^2)$ *and* $\Im(z^2)$? *(the real and imaginary components of* z^2*, respectively)*

b) *Evaluate* z^*z, *where* z^* *is the complex conjugate of* z?

c) *Write* z *in polar form.*

d) *Find* z^2 *using polar form.*

e) *Recall that* $\sqrt{z^*z}$ *is the* ***magnitude*** *of* z. *What is the magnitude of* z^2?

Problem 2 (20 points) *Let* $|a\rangle = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $|b\rangle = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, *and* $U = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$.

a) *Recall the inner product of two vectors* $|a\rangle, |b\rangle$ *is* $\langle a|b\rangle = \sum_i a_i^* b_i$. *Find* $\langle a|b\rangle$.

b) *Now apply* U *to* $|a\rangle$ *and* $|b\rangle$. *That is, evaluate* $|a'\rangle = U|a\rangle$ *and* $|b'\rangle = U|b\rangle$. *What is* $\langle a'|b'\rangle$?

c) *This is not a coincidence.* U *is a* ***unitary*** *matrix, which preserves the inner product. Equivalently, a unitary matrix satisfies* $U^\dagger U = I$ *(recall that* $(U^\dagger)_{ij} = U^*_{ji}$*). Show that the above matrix* U *satisfies this condition.*

d) *Now, let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a complex* 2×2 *matrix such that* $A^\dagger A = I$. *Show that* A *preserves the inner product between any two vectors* $|v\rangle, |w\rangle \in \mathbb{C}^2$. *(hint: write the vectors as a sum of orthonormal basis elements.)*

e) *Finally, show that any complex* $n \times n$ *matrix* M *satisfying* $M^\dagger M = I$ *preserves the inner product between any two vectors* $|v\rangle, |w\rangle \in \mathbb{C}^n$. *(hint: write the vectors as a sum of orthonormal basis elements, and use the summation definition of matrix multiplication.)*

Problem 3 (20 points) *Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ such that $|\alpha|^2 + |\beta|^2 = 1$. Recall the following definitions:*

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}, |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}.$$

a) *Recall from linear algebra that the projection of a vector $|v\rangle$ onto the subspace spanned by $|u\rangle$ is given by $proj_u(|v\rangle) = \frac{\langle u|v\rangle}{\langle u|u\rangle}|u\rangle$. What is the projection of $|\psi\rangle$ onto the subspace spanned by $|+\rangle$?*

b) *Find $P = |+\rangle\langle+|$. This is called the **outer product** and can be computed by standard matrix multiplication.*

c) *Find $P|\psi\rangle$.*

d) *Again, this is not a coincidence. In linear algebra we learned that the projection is a linear transformation, and therefore has a matrix representation. In this case, P is the **projection operator** onto the subspace spanned by $|+\rangle$. A projection operator satisfies $P^\dagger = P$ and $P^2 = P$. Verify that $|+\rangle\langle+|$, $|0\rangle\langle 0|$, and $|1\rangle\langle 1|$ all satisfy these conditions. (hint: you can solve this problem without writing out the matrices)*

Problem 4 (10 points) *An $n \times n$ square matrix M is called **positive semi-definite** (PSD), denoted $M \geq 0$, if $\langle x| M |x\rangle \geq 0$ for all vectors $|x\rangle \in C^n$.*

a) *Show that $|+\rangle\langle+|$ is PSD.*

b) *Show that all eigenvalues of a PSD matrix M are non-negative.*

Problem 5 (15 points) *Consider the following two orthonormal bases for $\mathbb{C}^2$: $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle, |-\rangle\}$.*

a) *For each orthonormal basis, show that the sum of the projectors onto the basis vectors equals the identity matrix. That is, show that $|0\rangle\langle 0| + |1\rangle\langle 1| = I$ and $|+\rangle\langle+| + |-\rangle\langle-| = I$.*

b) *Yet again, this is not a coincidence. This equality is captured by the **completeness relation**, which says that the projectors for a set of orthonormal basis elements sum to the identity. Prove the completeness relation for $\mathbb{C}^n$. (hint: Let $\{|v_j\rangle\}$ be an orthonormal basis for $\mathbb{C}^n$. How does $\sum_j |v_j\rangle\langle v_j|$ transform an arbitrary element of $\mathbb{C}$?)*

Problem 6 (20 points) *In Problem 3 we saw that projection operators satisfy $P^\dagger = P$ and $P^2 = P$. It turns out that projection operators are a special case of what are called **Hermitian** (or self-adjoint) matrices. A Hermitian matrix A satisfies $A^\dagger = A$.*

a) *Let $M = \begin{pmatrix}3 & 3+i\\ 3-i & 2\end{pmatrix}$. Show that M is Hermitian.*

b) *Find the eigenvalues and corresponding eigenvectors with length 1.*

For the next two parts of this problem, we need to recall the inner product of two vectors and their properties. Recall that for two vectors $|v\rangle, |w\rangle$ the inner product is $\langle v, w\rangle = \sum_i v_i^* w_i$. For any matrix and any pair of vectors, we have $\langle v, Aw\rangle = \langle A^\dagger v, w\rangle$. (You don't have to prove this, but we encourage you to do so). You need to use this relation in the following two parts.

c) *Show that a Hermitian matrix A has only real eigenvalues. (hint: write the eigenvector equation and use the above property of inner products).*

d) *Show that distinct eigenvalues of a Hermitian matrix correspond to orthogonal eigenvectors. You need to show that for any choice of eigenvalues $a \neq b$, the corresponding eigenvectors v_a, v_b fulfill the property $\langle v_a|v_b\rangle = 0$.*